User Manual for the
Bonadio-Huo-Levchenko-Pandalai-Nayar (BHLP)
Global Network Model
Version 1.0*

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1. Overview

This user manual accompanies the code for simulating the BHLP model. The user manual and the code are free and publicly available. Both are provided as a public service. We do not guarantee absence of errors in either this document or in the code itself. Please contact us with corrections and suggestions.

The BHLP model is built in successive steps in Huo, Levchenko, and Pandalai-Nayar (2019), Bonadio et al. (2021), and Bonadio et al. (2023). Please cite these three papers if you use the code in your own work.

The BHLP model is a multi-country, multi-sector general-equilibrium global network model with flexible labor supply, suitable to study international propagation of business cycle shocks. Notable features are (i) a general formulation of labor supply that nests both traditional trade assumptions (fixed aggregate labor supply, perfectly mobile across sectors), business cycle assumptions (flexible aggregate labor supply), as well as intermediate cases; and (ii) a nested CES formulation of final and intermediate goods bundles, that distinguishes between elasticities of substitution among inputs, and among source countries for the same input (Armington).

The model solution and code simulate the response of the world economy to shocks to (i) productivity; (ii) trade costs, (iii) tastes; (iv) trade imbalances. The shocks are specified by the user and can be customized to only occur in specific countries, sectors, or country-sector pairs.

The model is solved analytically to first order, and the accompanying code implements the analytical solution. We also present equations for, and provide the code for the exact solution using the hat algebra approach of Dekle, Eaton, and Kortum (2008).

The data requirements are a global input-output dataset such as the World Input-Output Database (Timmer et al., 2015) or the OECD Inter-Country Input-Output (ICIO) Database (OECD, 2022). The users can bring their own dataset to the code. We provide code to process the OECD ICIO database into the shape required to implement the model solution. This processing code includes options for the user to select the level of aggregation of sectors, and the set of countries to use. The data processing and manipulation code is in Stata.

The model solution code is in MATLAB. All structural parameters in the model can be set by the user (the code is pre-populated with "default" values that we use in our own calibrations). Below we list some examples of the memory requirements for running the code.

The rest of the user manual contains two sections. Section 2 presents the BHLP model and states the first-order analytical solution. Section 3 outlines the necessary data inputs and provides a user guide to running the code. The Appendix contains detailed derivations.

2. The BHLP Model

The model world economy is comprised of N countries indexed by n, m, and ℓ and J sectors indexed by j, i, and k. Each country n is populated by households that consume the final good available in country n and supply labor to firms.

2.1 Household Preferences

The representative household consists of a unit continuum of individuals indexed by ω. Each household can supply labor to any sector j with household-specific productivity $b_{nj}(\omega)$. If household ω decides work in sector j, it supplies $b_{nj}(\omega)H_n(\omega)$ effective units of labor and collects the labor income of $W_{nj}b_{nj}(\omega)H_n(\omega)$, where W_{nj} is the equilibrium price of one efficiency unit of labor in that country-sector. The household idiosyncratic labor productivity in sector j is distributed $b_{nj}(\omega) \sim \text{Fréchet}(\xi_{nj}, \mu)$, with dispersion parameter μ and central tendency parameter ξ_{nj} that can potentially vary by country and sector:

$$\Pr\left(b_{nj}(\omega) < b\right) = \exp\left(-\xi_{nj}b^{-\mu}\right).$$

Each individual chooses a sector in which to work, the number of hours worked, and the total consumption bundle to maximize the utility:

$$\max_{\mathcal{F}_n(\omega),H_n(\omega),j}\left(\mathcal{F}_n(\omega) - \chi_n\frac{H_n(\omega)^{1+1/\psi}}{1+1/\psi}\right) \tag{2.1}$$

subject to

$$P_n\mathcal{F}_n(\omega) = W_{nj}b_{nj}(\omega)H_n(\omega) + \sum_j R_{nj}K_{nj}(\omega) + D_n(\omega),$$

where $\mathcal{F}_n(\omega)$ is the the final consumption composite whose price index is given by P_n, $K_{nj}(\omega)$ is capital installed in sector j that is owned by household ω, earning R_{nj}, and $D_n(\omega)$ is the (exogenous) part of country n's trade deficit consumed by household ω. Total capital installed in j is $K_{nj} = \int_0^1 K_{nj}(\omega)d\omega$, and the aggregate deficit is $D_{nt} = \int_0^1 D_n(\omega)d\omega$. All the $K_{nj}(\omega)$'s and $D_n(\omega)$'s are exogenously given and fixed.

2.1.1 Sectoral Choice and Sectoral Effective Labor Supply

Conditional on working in sector j, household $\omega's$ maximization problem becomes, after substituting in the budget constraint:

$$\max_{H_n(\omega)} \left(\frac{W_{nj}b_{nj}(\omega)H_n(\omega)}{P_n} + \frac{\sum_j R_{nj}K_{nj}(\omega)}{P_n} + \frac{D_n(\omega)}{P_n} - \chi_n \frac{H_n(\omega)^{1+1/\psi}}{1+1/\psi} \right). \tag{2.2}$$

The first-order condition with respect to the hours worked yields the solution:

$$\frac{W_{nj}b_{nj}(\omega)}{P_n} - \chi_n H_n(\omega)^{1/\psi} = 0,$$

or

$$H_n(\omega) = \left(\frac{W_{nj}b_{nj}(\omega)}{\chi_n P_n} \right)^{\psi}. \tag{2.3}$$

Substitute the hours worked into the utility function to obtain the indirect utility for individual ω conditional on working in sector j:

$$U_j(\omega) = \left(\frac{1}{\psi+1} \right) \left(\frac{1}{\chi_n} \right)^{\psi} \left(\frac{W_{nj}b_{nj}(\omega)}{P_n} \right)^{1+\psi} + \frac{\sum_j R_{nj}K_{nj}(\omega)}{P_n} + \frac{D_n(\omega)}{P_n}. \tag{2.4}$$

Based on this vector of indirect utilities, household chooses sector j in which to work. Generically, household ω will choose to supply labor to only one sector. Appendix A.1 shows that the probability of choosing sector j is:

$$\pi_{nj}^H = \frac{\xi_{nj}(W_{nj})^{\mu}}{\sum_i \xi_{ni}(W_{ni})^{\mu}}. \tag{2.5}$$

Denote $W_n = \left(\sum_i \xi_{ni}(W_{ni})^{\mu} \right)^{\frac{1}{\mu}}$. Appendix section A.1 shows that the sectoral labor supply is:

$$H_{nj} = \xi_{nj} \left(\frac{1}{\chi_n} \frac{W_n}{P_n} \right)^{\psi} \left(\frac{W_{nj}}{W_n} \right)^{\mu-1} \Gamma \left(1 - \frac{\psi+1}{\mu} \right) \tag{2.6}$$

Denote by L_n the aggregate labor supply:

$$L_n = \left(\frac{1}{\chi_n} \frac{W_n}{P_n} \right)^{\psi} \tag{2.7}$$

up to the normalization constant $\Gamma \left(1 - \frac{\psi+1}{\mu} \right)$.

The aggregate labor supply elasticity is given by ψ. The labor supply elasticity to a given sector j conditional on a fixed aggregate labor supply is $\mu - 1$ (eq. 2.6). Lower values of μ imply less labor mobility across sectors.

Our specification nests a variety of labor supply frameworks in macro and trade. In canonical

trade models that reflect the long run, labor is assumed to be in fixed aggregate supply, but perfectly mobile across sectors. That would correspond to $\psi = 0$ and $\mu \to \infty$.

Increasingly, the trade literature has worked with models in which labor is not perfectly mobile across sectors, as exemplified for instance by the "Roy-Fréchet" framework (e.g. Lagakos and Waugh, 2013; Hsieh et al., 2019; Galle, Rodríguez-Clare, and Yi, 2023). These models would correspond to $\psi = 0$ and a finite μ set to the researcher's preferred value.

In canonical macro models that reflect adjustments of the economy to shocks at the business cycle frequency, aggregate labor supply is variable, $\psi > 0$. Variable aggregate labor supply can coexist with either perfect labor mobility across sectors ($\mu \to \infty$), or frictional mobility across sectors (finite μ), depending on one's preferred calibration.

2.1.2 Final Consumption

The final consumption bundle consists of two nests. The upper nest is a CES aggregate of sectoral bundles. In the lower nest, consumption of each sectoral bundle aggregates goods from all source countries. The upper nest is:

$$\mathcal{F}_n = \left[\sum_j \zeta_{nj}^{\frac{1}{\rho}} \mathcal{F}_{nj}^{\frac{\rho-1}{\rho}} \right]^{\frac{\rho}{\rho-1}}. \tag{2.8}$$

Sector j's bundle is an Armington aggregate of goods coming from different countries:

$$\mathcal{F}_{nj} = \left[\sum_m \mu_{mnj}^{\frac{1}{\gamma}} \mathcal{F}_{mnj}^{\frac{\gamma-1}{\gamma}} \right]^{\frac{\gamma}{\gamma-1}}. \tag{2.9}$$

Thus, ρ is the elasticity of substitution among different goods (e.g. food vs. clothing) in final use, and γ is the elasticity of substitution between different varieties of e.g. food coming from different source countries (the Armington elasticity).

2.2 Production

Firms are competitive. A representative firm in sector j in country n operates a CRS production function

$$Y_{nj} = Z_{nj} \left(K_{nj}^{\alpha_j} H_{nj}^{1-\alpha_j} \right)^{\eta_j} X_{nj}^{1-\eta_j}$$

where X_{nj} is the intermediate input bundle:

$$X_{nj} \equiv \left(\sum_i \vartheta_{i,nj}^{\frac{1}{\varepsilon}} X_{i,nj}^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}},$$

and in turn the intermediate input use of sector i is an aggregate from different source countries:

$$X_{i,nj} \equiv \left(\sum_m \mu_{mi,nj}^{\frac{1}{\nu}} X_{mi,nj}^{\frac{\nu-1}{\nu}} \right)^{\frac{\nu}{\nu-1}}.$$

Thus, similarly to the final consumption bundle, intermediate input usage is characterized by two CES nests: the upper nest that aggregates sectoral inputs (e.g. wheels, engines, bodies in the assembly of cars), governed by a substitution elasticity ε; and a lower nest that aggregates the sectoral input varieties coming from different countries (e.g. engines from the US, engines from Canada, engines from Mexico), governed by substitution elasticity ν (the Armington elasticity).

The final and intermediate elasticities in each nest can be set equal to each other, or to different values. For example, single-nest aggregation that pools all countries and sectors would set $\rho = \gamma$ and $\varepsilon = \nu$. If one does not want to make a distinction between substitution elasticities in final vs. intermediate use, one would set $\rho = \varepsilon$ and $\gamma = \nu$. The code is written so that the user has flexibility in setting all 4 of these elasticities independently.

2.3 Price Indices and Expenditure Shares

Let P_{mi} be the factory-gate price of good i in country m. Under the maintained assumption of perfect competition, it will equal the cost function of production in m, i.

Price indices. The ideal price index of sectoral bundle $\mathcal{F}_{nj}$ is:

$$P_{nj}^f = \left[\sum_m \mu_{mnj} (P_{mnj}^f)^{1-\gamma} \right]^{\frac{1}{1-\gamma}}, \tag{2.10}$$

where P_{mnj}^f is the price faced by final consumers in n of sector j goods coming from m. The ideal price index of overall final consumption $\mathcal{F}_n$ is:

$$P_n = \left[\sum_j \zeta_{nj} \left(P_{nj}^f \right)^{1-\rho} \right]^{\frac{1}{1-\rho}}. \tag{2.11}$$

The ideal price index of the sector i intermediate input bundle in (n, j), $X_{i,nj}$, is:

$$P_{i,nj}^X = \left[\sum_m \mu_{mi,nj} P_{mi,nj}^{1-\nu} \right]^{\frac{1}{1-\nu}},$$

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where $P_{mi,nj}$ is the price of (m, i) inputs faced by sector (n, j). The ideal price index of the overall intermediate input bundle X_{nj} is

$$P_{nj}^X = \left[\sum_i \vartheta_{i,nj} \left(P_{i,nj}^X \right)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}.$$

Trade costs. Trade in final goods is subject to iceberg costs $\tilde{\tau}_{mnj}^f$ to ship good j from country m to country n (throughout, we adopt the convention that the first subscript denotes source, and the second destination). Trade in intermediates is subject to iceberg costs $\tilde{\tau}_{mi,nj}^x$ to ship good i from country m to country n, sector j. We keep the notation for final and intermediate trade costs separate for the sake of generality, but users are free to equalize trade costs in final and intermediate use. No arbitrage in shipping implies that factory-gate prices and prices faced by consumers/firms are related by:

$$P_{mni}^f = \tilde{\tau}_{mni}^f P_{mi} \qquad P_{mi,nj} = \tilde{\tau}_{mi,nj}^x P_{mi}. \tag{2.12}$$

Shares. The final expenditure share for the sector j bundle in total final expenditure is:

$$\pi_{nj}^f = \frac{\zeta_{nj} \left(P_{nj}^f \right)^{1-\rho}}{\sum_k \zeta_{nk} \left(P_{nk}^f \right)^{1-\rho}}. \tag{2.13}$$

The expenditure share on goods from m in final expenditure on sector j is:

$$\pi_{mnj}^f = \frac{\mu_{mnj} \left(\tilde{\tau}_{mnj}^f P_{mj} \right)^{1-\gamma}}{\sum_\ell \mu_{\ell nj} \left(\tilde{\tau}_{\ell nj}^f P_{\ell j} \right)^{1-\gamma}}, \tag{2.14}$$

where we applied the no-arbitrage in shipping (2.12).

The share of sector i in total intermediate expenditure by (n, j) is:

$$\pi_{i,nj}^x = \frac{\vartheta_{i,nj} \left(P_{i,nj}^X \right)^{1-\varepsilon}}{\sum_k \vartheta_{k,nj} \left(P_{k,nj}^X \right)^{1-\varepsilon}}. \tag{2.15}$$

The share of intermediates from country m sector i in total intermediate spending by (n, j) on sector i is:

$$\pi_{mi,nj}^x = \frac{\mu_{mi,nj} \left(\tilde{\tau}_{mi,nj}^x P_{mi} \right)^{1-\nu}}{\sum_\ell \mu_{\ell i,nj} \left(\tilde{\tau}_{\ell i,nj}^x P_{\ell i} \right)^{1-\nu}}, \tag{2.16}$$

where once again we applied the no-arbitrage in shipping (2.12).

Firms' cost minimization implies:

$$W_{nj}H_{nj} = (1 - \alpha_j)\eta_j P_{nj}Y_{nj} \tag{2.17}$$

$$R_{nj}K_{nj} = \alpha_j \eta_j P_{nj}Y_{nj} \tag{2.18}$$

$$P_{mi,nj}X_{mi,nj} = \pi^x_{i,nj}\pi^x_{mi,nj}\left(1 - \eta_j\right)P_{nj}Y_{nj}. \tag{2.19}$$

2.4 Equilibrium

An equilibrium in this economy is a set of goods and factor prices $\{P_{nj}, W_{nj}, R_{nj}\}$, factor allocations $\{H_{nj}\}$, and goods allocations $\{Y_{nj}\}$, $\{\mathcal{F}_{mnj}, X_{mi,nj}\}$ for all countries and sectors such that (i) households maximize utility; (ii) firms maximize profits; and (iii) all markets clear. International trade is balanced in each country up to a vector of deficits D_m. Trade is balanced at the global level ($\sum_m D_m = 0$).

At the sectoral level, the following market clearing condition has to hold for each country n sector j:

$$P_{nj}Y_{nj} = \sum_m \left(\sum_i \eta_i P_{mi}Y_{mi} + D_m\right)\pi^f_{mj}\pi^f_{nmj} + \sum_m\sum_i (1 - \eta_i)P_{mi}Y_{mi}\pi^x_{j,mi}\pi^x_{nj,mi}. \tag{2.20}$$

Labor market clearing in each country n and sector j implies:

$$W_{nj}\xi_{nj}\left(\frac{1}{\chi_{nt}}\frac{W_n}{P_n}\right)^\psi \left(\frac{W_{nj}}{W_n}\right)^{\mu-1}\Gamma\left(1 - \frac{\psi + 1}{\mu}\right) = \left(1 - \alpha_j\right)\eta_j P_{nj}Y_{nj}. \tag{2.21}$$

Since K_{nj} is fixed, capital market clearing is given by (2.18).

2.5 Shocks

The code is written to subject the world economy to 5 vectors of shocks:

1. Country-sector productivity shocks (Z_{nj})

2. Country-sector taste shocks in final consumption (ζ_{nj}) and country-sector-by-sector taste shocks in intermediate use ($\vartheta_{i,nj}$)

3. Country-pair-by-sector final goods trade cost shocks (τ^f_{mni}), and country-sector-by-country-sector intermediate input trade cost shocks ($\tau^x_{mi,nj}$) (see note below)

4. Country-level deficit shocks (D_m).

The shocks are specified by the user. For instance, one might be interested in the global effects of a US aggregate productivity shock, or the impact of shutting off trade between two specific groups of countries.

Other shocks and isomorphisms. The impact on real value added of partner-specific bilateral taste shocks in the final usage μ_{mnj} and intermediate usage $\mu_{mi,nj}$ is isomorphic to the impact of trade cost shocks $\tilde{\tau}^f_{mni}$ and $\tilde{\tau}^x_{mi,nj}$, respectively. In fact, we define composite trade cost-cum-taste shocks, and the solution will be written in terms of these objects:

$$\tau^f_{mni} \equiv \mu^{\frac{1}{1-\gamma}}_{mnj} \tilde{\tau}^f_{mni} \quad \text{and} \quad \tau^x_{mi,nj} \equiv \mu^{\frac{1}{1-\nu}}_{mi,nj} \tilde{\tau}^x_{mi,nj}.$$

Country-sector-specific labor supply shocks can be modeled as shifts in ξ_{ni}. It turns out that in their impact on real value added these labor supply shocks are isomorphic to productivity shocks Z_{ni}, up to a scaling factor (Huo, Levchenko, and Pandalai-Nayar, 2019). Thus, we do not include separate code to compute the economy's response to these labor supply shocks.

2.6 Key Model Outcomes: GDP and Sectoral Real Value Added

The key objects that the code will output are changes in real GDP and changes in sector-country value added.

Real GDP. Following systems of national accounts (SNA) conventions, real GDP in levels is defined as output evaluated at base prices, minus inputs evaluated at base prices. In practice, we will be subjecting the economy to a vector of exogenous shocks, and will be computing log-changes in real GDP with respect to the pre-shock equilibrium. So effectively, the base prices are the pre-shock prices. Denote by a ′ (prime) the post-shock values of variables, and let the variables without primes denote pre-shock (base) values. Then real GDP following a shock is given by:

$$V'_n = \sum_{j=1}^{J} \left(P_{nj} Y'_{nj} - P^X_{nj} X'_{nj} \right). \tag{2.22}$$

Denote by a "ln" the log-deviation of a variable from its pre-shock value. The log change in real GDP following a shock is:

$$
\begin{aligned}
\ln V_n &= \sum_j \left(\frac{P_{nj} Y_{nj}}{V_n} \ln Y_{nj} - \frac{P^X_{nj} X_{nj}}{V_n} \ln X_{nj} \right) \\
&= \sum_j \frac{P_{nj} Y_{nj}}{V_n} \left(\ln Y_{nj} - \frac{P^X_{nj} X_{nj}}{P_{nj} Y_{nj}} \ln X_{nj} \right) \\
&= \sum_j \frac{P_{nj} Y_{nj}}{V_n} \left(\ln Z_{nj} + \eta_j (1 - \alpha_j) \ln H_{nj} + (1 - \eta_j) \ln X_{nj} - \frac{P^X_{nj} X_{nj}}{P_{nj} Y_{nj}} \ln X_{nj} \right) \\
&= \sum_{j=1} \frac{P_{nj} Y_{nj}}{V_n} \ln Z_{nj} + \sum_{j=1} \eta_j (1 - \alpha_j) \frac{P_{nj} Y_{nj}}{V_n} \ln H_{nj}, \tag{2.23}
\end{aligned}
$$

where the last line comes from the fact that $\frac{P_{nj}^X X_{nj}}{P_{nj} Y_{nj}} = (1 - \eta_j)$. The weight $\frac{P_{nj} Y_{nj}}{V_n}$ is the fraction of sectoral gross output in aggregate value added, otherwise known as the Domar weight. These weights are pre-shock. The real GDP outputted by the model solution code following any vector of shocks is the object in (2.23).

Sectoral Real Value Added. Similarly, sectoral real value added is defined as gross output evaluated at base prices minus input usage evaluated at base prices. Thus the post-shock level of sectoral value added is defined as:

$$V'_{nj} = P_{nj} Y'_{nj} - P_{nj}^X X'_{nj}.$$

The log-deviation of real sectoral value added is given by:

$$\ln V_{nj} = \frac{P_{nj} Y_{nj}}{V_{nj}} \left(\ln Z_{nj} + \eta_j (1 - \alpha_j) \ln H_{nj} \right)$$

$$= \frac{1}{\eta_j} \ln Z_{nj} + (1 - \alpha_j) \ln H_{nj}, \tag{2.24}$$

where the second line comes from the fact that $\frac{V_{nj}}{P_{nj} Y_{nj}} = \eta_j$. The real sectoral value added outputted by the model solution code following any vector of shocks is the object in (2.24).

It is easy to verify that (2.23) and (2.24) respect the accounting identity that the real GDP of the economy is simply the sum of the real sectoral value addeds:

$$V'_n = \sum_j V'_{nj},$$

so that the log change in real GDP is equal to the value-added-share-weighted log change in sectoral real value added:

$$\ln V_n = \sum_j \frac{V_{nj}}{V_n} \ln V_{nj}. \tag{2.25}$$

2.7 Influence Matrix (Linearized Solution)

We now provide a first-order analytical solution for the response of the economy to the shocks. The solution is first-order so that it comes from linearizing the model around the pre-shock equilibrium. The solution expresses the vector of log changes in objects of interest like output, hours and value added as a product of a vector of the shocks provided by the user and influence matrices that translate the shocks to equilibrium responses of the variables of interest. As above, we denote by a "ln" the log-deviation of a variable from its pre-shock value. Bold font is used to denote vectors or matrices.

Market clearing conditions. Under an exogenous trade deficit, the linearized market clearing conditions (2.20) become:

$$
\ln P_{nj} + \ln Y_{nj} = \sum_m \sum_i \frac{\eta_i P_{mi} Y_{mi}}{P_m \mathcal{F}_m} \frac{\pi^f_{mj} \pi^f_{nmj} P_m \mathcal{F}_m}{P_{nj} Y_{nj}} \left(\ln \pi^f_{mj} + \ln \pi^f_{nmj} + \ln P_{mi} + \ln Y_{mi} \right)
$$

$$
+ \sum_m \frac{\pi^f_{mj} \pi^f_{nmj} P_m \mathcal{F}_m}{P_{nj} Y_{nj}} \frac{D_m WGDP}{P_m \mathcal{F}_m} \left(\Delta + \sum_o \sum_k \frac{\eta_k P_{ok} Y_{ok}}{WGDP} \left(\ln P_{ok} + \ln Y_{ok} \right) \right) \qquad (2.26)
$$

$$
+ \sum_m \sum_i (1 - \eta_i) \frac{P_{mi} Y_{mi}}{P_{nj} Y_{nj}} \pi^x_{j,mi} \pi^x_{nj,mi} \left(\ln \pi^x_{j,mi} + \ln \pi^x_{nj,mi} + \ln P_{mi} + \ln Y_{mi} \right)
$$

where $\Delta \equiv \hat{D}_m - 1 = \frac{D'_m - D_m}{D_m}$ is the proportional change in D_m that can accommodate potentially negative values of trade deficits, and the trade deficits D_m are expressed as share of world GDP.

Expenditure shares. Linearized final sectoral consumption and international trade shares (2.13)-(2.14) are:

$$
\ln \pi^f_{nj} = \ln \zeta_{nj} + (1 - \rho) \sum_m \pi^f_{mnj} \left(\ln \tau^f_{mnj} + \ln P_{mj} \right) \qquad (2.27)
$$

$$
- \sum_k \pi^f_{nk} \ln \zeta_{nk} - (1 - \rho) \sum_k \pi^f_{nk} \left[\sum_m \pi^f_{mnk} \left(\ln \tau^f_{mnk} + \ln P_{mk} \right) \right]
$$

and

$$
\ln \pi^f_{mnj} = (1 - \gamma) \left(\ln \tau^f_{mnj} + \ln P_{mj} - \sum_o \pi^f_{onj} \left(\ln \tau^f_{onj} + \ln P_{oj} \right) \right). \qquad (2.28)
$$

Linearized intermediate sectoral input use and international trade shares (2.15)-(2.16) are

$$
\ln \pi^x_{i,nj} = \ln \vartheta_{i,nj} + (1 - \varepsilon) \left(\sum_m \pi^x_{mi,nj,t} \left(\ln \tau^x_{mi,nj} + \ln P_{mi} \right) \right) \qquad (2.29)
$$

$$
- \sum_k \pi^x_{k,nj,t} \ln \vartheta_{k,nj} - (1 - \varepsilon) \sum_k \pi^x_{k,nj,t} \sum_m \pi^x_{mk,nj,t} \left(\ln \tau^x_{mk,nj} + \ln P_{mk} \right)
$$

and

$$
\ln \pi^x_{mi,nj} = (1 - \nu) \left(\ln \tau^x_{mi,nj} + \ln P_{mi} - \sum_k \pi^x_{ki,nj} \left(\ln \tau^x_{ki,nj} + \ln P_{ki} \right) \right). \qquad (2.30)
$$

Prices. Starting from the fact that the price is proportional to marginal cost, after substituting wages and rental rate of capital from firms' optimality conditions (2.17)-(2.18), noting that sectoral capital is fixed, and taking logs, we get:

$$
\ln P_{nj} = - \ln Z_{nj} + \eta_j \left(\ln P_{nj} + \ln Y_{nj} \right) - (1 - \alpha_j) \eta_j \ln H_{nj} + (1 - \eta_j) \ln P^X_{nj}. \qquad (2.31)
$$

Linearized price index changes are:

$$\ln P_{nj}^X = \sum_i \pi_{i,nj}^x \left[\frac{\ln \vartheta_{i,nj}}{1-\varepsilon} + \sum_m \pi_{mi,nj}^x \left(\ln \tau_{mi,nj}^x + \ln P_{mi} \right) \right] \tag{2.32}$$

$$\ln P_n = \sum_k \pi_{nk}^f \left[\frac{1}{1-\rho} \ln \zeta_{nk} + \sum_m \pi_{mnk}^f \left(\ln \tau_{mnk}^f + \ln P_{mk} \right) \right]. \tag{2.33}$$

Labor hours. The log change in hours is:

$$\ln H_{nj} = \ln L_n + \frac{\mu - 1}{\mu} \ln \pi_{nj}^H. \tag{2.34}$$

Linearized aggregate hours and aggregate wage index are (the model does not consider shocks to ξ and χ):

$$\frac{1}{\psi} \ln L_n = \ln W_n - \ln P_n \tag{2.35}$$

where

$$\ln W_n = \sum \pi_{nj}^H \ln W_{nj}. \tag{2.36}$$

The logged versions of (2.5) and (2.17) are:

$$\ln W_{nj} = \ln P_{nj} + \ln Y_{nj} - \ln H_{nj} \tag{2.37}$$

and

$$\ln \pi_{nj}^H = \mu \ln W_{nj} - \mu \ln W_n. \tag{2.38}$$

Influence matrices. Appendix A.2 details the derivation of the analytical solutions to the response of output, sectoral hours, and aggregate labor to all the shocks. The solutions take the following forms:

$$\ln \mathbf{Y} = \mathbf{\Lambda}_Z^Y \ln \mathbf{Z} + \mathbf{\Lambda}_\zeta^Y \ln \zeta + \mathbf{\Lambda}_{\tau^f}^Y \ln \tau^f + \mathbf{\Lambda}_\vartheta^Y \ln \vartheta + \mathbf{\Lambda}_{\tau^x}^Y \ln \tau^x + \mathbf{\Lambda}_\Delta^Y \mathbf{\Delta} \tag{2.39}$$

$$\ln \mathbf{H} = \mathbf{\Lambda}_Z^H \ln \mathbf{Z} + \mathbf{\Lambda}_\zeta^H \ln \zeta + \mathbf{\Lambda}_{\tau^f}^H \ln \tau^f + \mathbf{\Lambda}_\vartheta^H \ln \vartheta + \mathbf{\Lambda}_{\tau^x}^H \ln \tau^x + \mathbf{\Lambda}_\Delta^H \mathbf{\Delta}. \tag{2.40}$$

The matrices are computable using observed shares and given the structural parameters. The code computes all the influence matrices that define these solutions.

Sectoral real value added and real GDP. In matrix form, equation (2.24) becomes:

$$\ln \mathbf{V} = \boldsymbol{\eta}^{-1} \ln \mathbf{Z} + (\mathbf{I} - \boldsymbol{\alpha}) \ln \mathbf{H}.$$

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Plugging in for changes in $\ln \mathbf{H}$ due to exogenous shocks using (2.40), we get:

$$\ln \mathbf{V} = \Lambda_Z^V \ln \mathbf{Z} + \Lambda_\zeta^V \ln \zeta + \Lambda_{\tau^f}^V \ln \tau^f + \Lambda_\vartheta^V \ln \vartheta + \Lambda_{\tau^x}^V \ln \tau^x + \Lambda_\Delta^V \Delta, \tag{2.41}$$

where $\Lambda_Z^V = \eta^{-1} + (\mathbf{I} - \alpha) \Lambda_Z^H$ and $\Lambda_s^V = (\mathbf{I} - \alpha) \Lambda_s^H$ for $s \in \{\zeta, \tau^f, \vartheta, \tau^x, \Delta\}$.

Real GDP growth $\ln V_n$ is given by the weighted growth in sector-level real value added growth, (2.25). Hence we define an $N \times NJ$ matrix $\mathbf{A}$ such that $\mathbf{A}_{n,nj} = \frac{V_{nj}}{V_n}$ and $\mathbf{A}_{n,mj} = 0$ for $n \neq m$. Then the impact of any shock s on the vector of countries' real GDPs is given by $\mathbf{A}\Lambda_s^V$, $s \in \{Z, \zeta, \tau^f, \vartheta, \tau^x, \Delta\}$.

2.8 Exact Solution

The response of the model economy to a vector of shocks can also be obtained via the exact hat algebra approach of Dekle, Eaton, and Kortum (2008). This approach writes the proportional change in the equilibrium objects as a function of the pre-shock expenditure and output shares and the proportional changes in shocks. The result is a system of non-linear equations that is solved numerically. The nonlinear equations and the solution algorithm are summarized in Appendix A.3, and implemented in the code as described below.

3. Code Manual

3.1 Items To Be Specified by the User

In order to run the code, the user should specify the following items:

1. The level of aggregation of sectors and countries. This is accomplished by manipulating the Excel file `Data Cleaning/sector_aggregation_alpha.xlsx`. In the "`sectors`" tab, the user should edit column D to provide an aggregation of the original OECD ICIO sectors into sectors to be used in the code. In the "`countries`" tab, the user should manipulate column B to place countries into the ROW ("Rest of the World") category.

2. The shocks to which the world economy will be subjected. This is accomplished in the Excel file `Data Cleaning/shocks_feed.xlsx`. The file contains the tabs for each shock. The shocks should be entered in log-deviation (e.g. 0.01 for an approximate 1% shock), except for the deficit shock that should be entered as a gross growth rate (e.g. 1 for no change, 0 to eliminate deficits).

3. Structural elasticities, listed in Table 1. These can be set in `Matlab/configuration.m`. That file is pre-populated with "default" values. Note that our baseline choice of α_j is 0 (an "equipped-labor" model). However, the data construction code outlined below produces values of α_j consistent with any desired sectoral aggregation as an alternative. The file `sector_aggregation_alpha.xlsx` is pre-populated with default values of α_j, but can also be

Table 1: Model Parameters

Parameter	Related to
ρ	final cross-sector substitution elasticity
ε	intermediate cross-sector subst. elasticity
γ	Armington elasticity in final consumption
ν	Armington elasticity in intermediate inputs
ψ	Frisch elasticity of labor supply
μ	sectoral labor supply elasticity
η_j	value added share in gross output
α_j	capital share in value added

Table 2: Expenditure Shares

Share	Description
π^f_{njt}	sectoral consumption shares
π^f_{mnj}	trade shares in final trade
$\pi^x_{i,nj}$	sectoral intermediate use
$\pi^x_{mi,nj}$	trade shares in sectoral intermediate use

used to set one's own preferred values of the capital share.[1]

3.2 Data Preparation

The MATLAB codes require the following two types of data in matrix form, summarized in Table 2:

1. Final consumption data at the country-sector level.

   - Variables are ordered as follows: export country (string), import country (string), sector name (string), year (numeric), value of trade flow (numeric), and sector number (numeric).

   - Data are sorted according to the order: export country, import country, sector, year.

   - When read into MATLAB, the data are read as a structured array with three columns of data.

2. Intermediate trade data at the country-sector pair level.

   - Variables are ordered as follows: export country (string), export sector (string), import country (string), import sector (string), year (numeric), value of trade flow (numeric), export sector number (numeric), import sector number (numeric).

   - Data are sorted according to the order: export country, export sector, import country, import sector, year.

---

[1]For convenience, the Excel sheet provides values of α_j computed from the WIOD Social Economics Accounts, 2016 release. See the subfolder `WIOD_DATA` for details.

- When read into MATLAB, the data are read as a structured array with four columns of data.

We provide the user with a default dataset, taken from the 2021 edition of the OECD Inter-Country Input-Output (ICIO) Tables, downloaded in July 2023.[2] The cleaning codes make the following baseline adjustments to the OECD data:

1. Add up CN1 and CN2 to be CHN

2. Add up MX1 and MX2 to be MEX

3. Add up all absorption into a single "FINAL" sector

4. To deal with the issue that some country-sector pairs have a 0 value of trade flows, the code assigns a small value (the minimum of observed positive values) to the observed zeros when processing the data, to ensure all the matrix operations in the model can be computed.[3]

3.3 Running the Code

After specifying the user options as described in Section 3.1, the following files should be run:

1. From the `Data Cleaning` directory, run in Stata `do1_OECD_data_clean.do` (optional, as we provide the output of that file in the "temp_files" subfolder), `do2_OECD_data_aggregation.do`, and `do3_prepare_shocks.do`. This processes the OECD data from 1995 to 2018 into the consistent format outlined above. The output is saved in a subfolder called "matlab_input_data" where it can be accessed by the main model code.

2. From the `Matlab` directory, run

   (a) `run0_buildSS.m`: Create a wedge-free steady-state dataset using the raw trade data, using exact-hat algebra.

   (b) `run1_compute_influence.m`: Compute the influence matrices, as described in Section 2.7, whose formulas are given in Appendix A.2. For ease of reading the manual and the code together, we tried to label the matrices with the same names in the manual and the MATLAB code.

   (c) `run2_solveshocks.m`: Compute the change in sectoral output, sectoral value added, and GDP following a vector of shocks.

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[2]Since July 2023, the OECD has updated the ICIO tables with additional countries. See the webarchive version we used here: https://web.archive.org/web/20230713114340/https://www.oecd.org/sti/ind/inter-country-input-output-tables.htm

[3]While the model doesn't require all pairs to be positive to be solved, the algorithm encounters problems if a specific country-sector has 0 total output. Changing all pairs to small positive numbers avoids this issue while making no quantitative difference.

Table 3: Runtime and memory

Countries	Sectors	Runtime (minutes)	Peak memory (GB)
29	24	12.3	3.14
29	45	99.1	18.4
37	45	231.3	38.0

Note: all computations performed on a Windows 10 enterprise machine with Intel Core i9 3.60 GHz Processor CPU and 128 GB RAM. The tables display the runtime and peak memory required to run the files run0 (computing the wedge-free dataset) and run1 (computing the influence vector) of the matlab code. The exact-hat algebra algorithm's convergence time depends on the exogenous shocks.

To obtain instead the exact hat algebra solution, run `run3_exacthat.m`. That file implements the solution algorithm described in Section 2.8 and Appendix A.3, by calling the function `solve_hat.m`.

The final output is exported to the `final_output.xlsx` Excel file, which outputs the changes in GDP, aggregate L, sectoral gross output (Y), value added (V) and hours (H).

Note that the memory requirements for running the MATLAB model code with the full set of countries (60) and sectors (45) in the OECD data are onerous and outside the scope of most machines. This is why we provide a simple customizable aggregation code described in Section 3.1 where users can choose the set of countries and sectors they are interested in for the baseline analysis and group the residual countries/sectors into aggregates. The tradeoff is that more sectors and/or countries will increase the level of resolution of the results, but will require more memory. Table 3 outlines the runtime on machines with varied RAM of different combinations of countries and sectors to illustrate this tradeoff.

A. Model Appendix

A.1 Detailed Derivations of Sectoral Choice and Sectoral Effective Labor Supply

The indirect utility conditional on working in j (2.4), ignoring the constant terms that do not depend on household choices, becomes:

$$U_j(\omega) = \left(\frac{1}{\psi+1}\right)\left(\frac{1}{\chi_n}\right)^{\psi}\left(\frac{W_{nj}b_{nj}(\omega)}{P_n}\right)^{1+\psi}. \tag{A.1}$$

Since the household chooses the sector that gives the largest indirect utility, the probability of choosing sector j is:

$$\Pr\left[U_j(\omega) > U_i(\omega), \forall i \neq j\right] = \Pr\left[\left(W_{nj}b_{nj}(\omega)\right)^{1+\psi} > \left(W_{ni}b_{ni}(\omega)\right)^{1+\psi}, \forall i \neq j\right].$$

The analytical form of this probability is derived using the Fréchet distributional assumption on $b_{nj}(\omega)$. Re-arranging the expression for the choice probability, conditional on $b_{nj}(\omega)$, the probability is:

$$\Pr\left[U_j(\omega) > U_i(\omega), \forall i \neq j | b_{nj}(\omega)\right] = \Pr\left[b_{ni}(\omega) < \frac{W_{nj}}{W_{ni}}b_{nj}(\omega)|b_{nj}(\omega)\right]$$

$$= \exp\left(-\sum_{i\neq j}\xi_{ni}\left(\frac{W_{nj}}{W_{ni}}\right)^{-\mu}\left(b_{nj}(\omega)\right)^{-\mu}\right).$$

Since $F(b) = \exp\left(-\xi_{nj}b^{-\mu}\right)$, then $dF(b) = \mu\xi_{nj}b^{-\mu-1}\exp\left(-\xi_{nj}b^{-\mu}\right)$, integrate out $b_{nj}(\omega)$:

$$\Pr\left[U_j(\omega) > U_i(\omega), \forall i \neq j\right] = \int \exp\left(-\sum_{i\neq j}\xi_{ni}\left(\frac{W_{nj}}{W_{ni}}\right)^{-\mu}(b)^{-\mu}\right)\mu\xi_{nj}b^{-\mu-1}\exp\left(-\xi_{nj}b^{-\mu}\right)db$$

$$= \left[\frac{\xi_{nj}}{\sum_i \xi_{ni}\left(\frac{W_{nj}}{W_{ni}}\right)^{-\mu}}\exp\left(-\sum_i \xi_{ni}\left(\frac{W_{nj}}{W_{ni}}\right)^{-\mu}(b)^{-\mu}\right)\right]_0^{\infty},$$

which leads to the probability of choosing sector j is given by (2.5). The total effective labor supply to sector j is:

$$H_{nj} = \pi_{nj}^H \int_{\omega\in j} H_n(\omega)b_{nj}(\omega)d\omega = \pi_{nj}^H E\left[H_n(\omega)b_{nj}(\omega)|j\right]. \tag{A.2}$$

Plug in the expression of $H_n(\omega)$:

$$E\left[H_n\left(\omega\right)b_{nj}\left(\omega\right)|j\right] = E\left[\left(\frac{1}{\chi_n}\frac{1}{P_n}\right)^{\psi}\left(W_{nj}\right)^{-1}\left(W_{nj}b_{nj}\left(\omega\right)\right)^{1+\psi}\mid\left(W_{nj}b_{nj}\left(\omega\right)\right)^{1+\psi} > \left(W_{ni}b_{ni}\left(\omega\right)\right)^{1+\psi}\forall i\right]$$

$$= \left(\frac{1}{\chi_n}\frac{1}{P_n}\right)^{\psi}\left(W_{nj}\right)^{-1}E\left[\left(W_{nj}b_{nj}\left(\omega\right)\right)^{1+\psi}\mid\left(W_{nj}b_{nj}\left(\omega\right)\right)^{1+\psi} > \left(W_{ni}b_{ni}\left(\omega\right)\right)^{1+\psi}\forall i\right].$$

$$(A.3)$$

To evaluate the expectation, use the distributional assumption on $b_{nj}(\omega)$:

$$\Pr\left[b_{nj}\left(\omega\right) < b|\left(W_{nj}b_{nj}\left(\omega\right)\right)^{1+\psi} > \left(W_{ni}b_{ni}\left(\omega\right)\right)^{1+\psi}\forall i\right] = \frac{\Pr\left[b_{nj}\left(\omega\right) < b \cap \left(W_{nj}b_{nj}\left(\omega\right)\right)^{1+\psi} > \left(W_{ni}b_{ni}\left(\omega\right)\right)^{1+\psi}\forall i\right]}{\Pr\left[\left(W_{nj}b_{nj}\left(\omega\right)\right)^{1+\psi} > \left(W_{ni}b_{ni}\left(\omega\right)\right)^{1+\psi}\forall i\right]}.$$

By definition, the denominator is the probability of choosing sector j, and the numerator is the following integral:

$$\Pr\left[b_{nj}\left(\omega\right) < b|\left(W_{nj}b_{nj}\left(\omega\right)\right)^{1+\psi} > \left(W_{ni}b_{ni}\left(\omega\right)\right)^{1+\psi}\forall i\right] = \frac{\int_0^b\Pr\left[\left(W_{nj}s\right)^{1+\psi} > \left(W_{ni}b_{ni}\left(\omega\right)\right)^{1+\psi}\forall i\right]dF(s)}{\pi_{nj}^H}.$$

Evaluate the integral in the numerator:

$$\int_0^b\Pr\left[\left(W_{nj}s\right)^{1+\psi} > \left(W_{ni}b_{ni}\left(\omega\right)\right)^{1+\psi}\forall i\right]dF(s) = \int_0^b\Pr\left[b_{ni}\left(\omega\right) < \frac{W_{nj}s}{W_{ni}}\forall i\right]dF(s)$$

$$= \int_0^b\exp\left(-\sum_{i\neq j}\xi_{ni}\left(\frac{W_{nj}s}{W_{ni}}\right)^{-\mu}\right)dF(s)$$

$$= \int_0^b\exp\left(-\sum_{i\neq j}\xi_{ni}\left(\frac{W_{nj}s}{W_{ni}}\right)^{-\mu}\right)\mu\xi_{nj}s^{-\mu-1}\exp\left(-\xi_{nj}s^{-\mu}\right)ds$$

$$= \frac{\xi_{nj}}{\sum_i\xi_{ni}\left(\frac{W_{nj}}{W_{ni}}\right)^{-\mu}}\exp\left(-\sum_i\xi_{ni}\left(\frac{W_{nj}}{W_{ni}}\right)^{-\mu}b^{-\mu}\right).$$

Combine the numerator and denominator:

$$\Pr\left[b_{nj}\left(\omega\right) < b|\left(W_{nj}b_{nj}\left(\omega\right)\right)^{1+\psi} > \left(W_{ni}b_{ni}\left(\omega\right)\right)^{1+\psi}\forall i\right] = \exp\left(-\sum_i\xi_{ni}\left(\frac{W_{nj}}{W_{ni}}\right)^{-\mu}b^{-\mu}\right).$$

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Go back to computing the expectation:

$$E\left[\left(b_{nj}\left(\omega\right)\right)^{1+\psi}\mid\left(W_{nj}b_{nj}\left(\omega\right)\right)^{1+\psi}>\left(W_{ni}b_{ni}\left(\omega\right)\right)^{1+\psi}\forall i\right]=$$

$$\int_{o}^{\infty}b^{1+\psi}\exp\left(-\sum_{i}\xi_{ni}\left(\frac{W_{nj}}{W_{ni}}\right)^{-\mu}b^{-\mu}\right)\left(\sum_{i}\xi_{ni}\left(\frac{W_{nj}}{W_{ni}}\right)^{-\mu}\right)\mu b^{-\mu-1}db.$$

Re-arrange to obtain:

$$E\left[\left(b_{nj}\left(\omega\right)\right)^{1+\psi}\mid\left(W_{nj}b_{nj}\left(\omega\right)\right)^{1+\psi}>\left(W_{ni}b_{ni}\left(\omega\right)\right)^{1+\psi}\forall i\right]=$$

$$\left(\sum_{i}\xi_{ni}\left(\frac{W_{nj}}{W_{ni}}\right)^{-\mu}\right)\int_{0}^{\infty}\exp\left(-\sum_{i}\xi_{ni}\left(\frac{W_{nj}}{W_{ni}}\right)^{-\mu}b^{-\mu}\right)\mu b^{\psi-\mu}db.$$

Evaluating the integral yields:

$$E\left[\left(b_{nj}\left(\omega\right)\right)^{1+\psi}\mid\left(W_{nj}b_{nj}\left(\omega\right)\right)^{1+\psi}>\left(W_{ni}b_{ni}\left(\omega\right)\right)^{1+\psi}\forall i\right]=\left(W_{nj}\right)^{-(1+\psi)}\left[\sum_{i}\xi_{ni}\left(W_{ni}\right)^{\mu}\right]^{\frac{1+\psi}{\mu}}\Gamma\left(1-\frac{\psi+1}{\mu}\right)$$

where Γ is the gamma function. Plug into equation (A.3):

$$E\left[H_{n}\left(\omega\right)b_{nj}\left(\omega\right)\mid j\right]=\left(\frac{1}{\chi_{n}}\frac{1}{P_{n}}\right)^{\psi}\left(W_{nj}\right)^{-1}\left[\sum_{i}\xi_{ni}\left(W_{ni}\right)^{\mu}\right]^{\frac{1+\psi}{\mu}}\Gamma\left(1-\frac{\psi+1}{\mu}\right).$$

Denote $W_{n}=\left(\sum_{i}\xi_{ni}\left(W_{ni}\right)^{\mu}\right)^{\frac{1}{\mu}}$. Thus the sectoral labor is:

$$H_{nj}=\xi_{nj}\left(\frac{1}{\chi_{n}}\frac{1}{P_{n}}\right)^{\psi}\left(W_{nj}\right)^{\mu-1}\left[\sum_{i}\xi_{ni}\left(W_{ni}\right)^{\mu}\right]^{\frac{1+\psi}{\mu}-1}\Gamma\left(1-\frac{\psi+1}{\mu}\right)$$

$$=\xi_{nj}\left(\frac{1}{\chi_{n}}\frac{W_{n}}{P_{n}}\right)^{\psi}\left(\frac{W_{nj}}{W_{n}}\right)^{\mu-1}\Gamma\left(1-\frac{\psi+1}{\mu}\right),$$

which is equation (2.6) in the main text. Finally, the aggregate labor supply is obtained through adding up all the sectors. The total labor compensation is:

$$\sum_{j}H_{nj}W_{nj}=\sum_{j}\left(\frac{1}{\chi_{n}}\frac{W_{n}}{P_{n}}\right)^{\psi}\left(\frac{W_{nj}}{W_{n}}\right)^{\mu-1}\Gamma\left(1-\frac{\psi+1}{\mu}\right)\xi_{nj}W_{nj}$$

$$=\Gamma\left(1-\frac{\psi+1}{\mu}\right)\left(\frac{1}{\chi_{n}}\frac{W_{nt}}{P_{n}}\right)^{\psi}W_{n},$$

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implying that aggregate labor supply is:

$$L_n = \left(\frac{1}{\chi_n}\frac{W_n}{P_n}\right)^{\psi},$$

up to the normalization constant $\Gamma\left(1 - \frac{\psi+1}{\mu}\right)$, which is equation (2.7) in the main text.

A.2 Derivations for Influence Matrices

A.2.1 Market Clearing Conditions and Shares

Plugging (2.27), (2.28), (2.29), and (2.30) into (2.26):

$$
\begin{aligned}
\ln P_{nj} + \ln Y_{nj} &\approx \sum_m \sum_i \frac{\eta_i P_{mi} Y_{mi}}{P_m F_m} \frac{\pi^f_{mj}\pi^f_{nmj}P_m F_m}{P_{nj}Y_{nj}}\left(\ln P_{mi} + \ln Y_{mi}\right)\\
&+ \sum_m \frac{\pi^f_{mj}\pi^f_{nmj}P_m \mathcal{F}_m}{P_{nj}Y_{nj}}\left(\ln \zeta_{mj} + (1-\rho)\sum_o \pi^f_{omj}\left(\ln \tau^f_{omj} + \ln P_{oj}\right)\right)\\
&+ \sum_m \frac{\pi^f_{mj}\pi^f_{nmj}P_m \mathcal{F}_m}{P_{nj}Y_{nj}}\left(-\sum_k \pi^f_{mk}\ln\zeta_{mk} - (1-\rho)\sum_k \pi^f_{mk}\left[\sum_o \pi^f_{omk}\left(\ln\tau^f_{omk} + \ln P_{ok}\right)\right]\right)\\
&+ \sum_m \frac{\pi^f_{mj}\pi^f_{nmj}P_m\mathcal{F}_m}{P_{nj}Y_{nj}}\left((1-\gamma)\left(\ln\tau^f_{nmj} + \ln P_{nj} - \sum_o \pi^f_{omj}\left(\ln\tau^f_{omj} + \ln P_{oj}\right)\right)\right)\\
&+ \sum_m \frac{\pi^f_{mj}\pi^f_{nmj}P_m\mathcal{F}_m}{P_{nj}Y_{nj}}\frac{D_m WGDP}{P_m\mathcal{F}_m}\left(\left(\hat{D}_m - 1\right) + \sum_o\sum_k \frac{\eta_k P_{ok}Y_{ok}}{WGDP}\left(\ln P_{ok} + \ln Y_{ok}\right)\right)\\
&+ \sum_m\sum_i (1-\eta_i)\frac{P_{mi}Y_{mi}}{P_{nj}Y_{nj}}\pi^x_{j,mi}\pi^x_{nj,mi}\left(\left(\ln\vartheta_{j,mi}\right) + (1-\varepsilon)\left(\sum_o \pi^x_{oj,mi}\left(\ln\tau^x_{oj,mi} + \ln P_{oj}\right)\right)\right)\\
&+ \sum_m\sum_i (1-\eta_i)\frac{P_{mi}Y_{mi}}{P_{nj}Y_{nj}}\pi^x_{j,mi}\pi^x_{nj,mi}\left(-\sum_k \pi^x_{k,mi}\ln\vartheta_{k,mi}\right)\\
&+ \sum_m\sum_i (1-\eta_i)\frac{P_{mi}Y_{mi}}{P_{nj}Y_{nj}}\pi^x_{j,mi}\pi^x_{nj,mi}\left(-(1-\varepsilon)\sum_k \pi^x_{k,mi,t}\sum_o \pi^x_{ok,mi}\left(\ln\tau^x_{ok,mi} + \ln P_{ok}\right)\right)\\
&+ \sum_m\sum_i (1-\eta_i)\frac{P_{mi}Y_{mi}}{P_{nj}Y_{nj}}\pi^x_{j,mi}\pi^x_{nj,mi}\left((1-\nu)\left(\ln\tau^x_{nj,mi} + \ln P_{nj}\right)\right)\\
&+ \sum_m\sum_i (1-\eta_i)\frac{P_{mi}Y_{mi}}{P_{nj}Y_{nj}}\pi^x_{j,mi}\pi^x_{nj,mi}\left((1-\nu)\left(-\sum_o \pi^x_{oj,mi}\left(\ln\tau^x_{oj,mi} + \ln P_{oj}\right)\right)\right)\\
&+ \sum_m\sum_i (1-\eta_i)\frac{P_{mi}Y_{mi}}{P_{nj}Y_{nj}}\pi^x_{j,mi}\pi^x_{nj,mi}\left(\ln P_{mi} + \ln Y_{mi}\right).
\end{aligned}
$$

Re-arrange and collect terms:

$$\ln P_{nj} + \ln Y_{nj} \approx [\sum_m \sum_i \frac{\eta_i P_{mi} Y_{mi}}{P_m F_m} \frac{\pi^f_{mj} \pi^f_{nmj} P_m F_m}{P_{nj} Y_{nj}} + \sum_m \sum_i (1 - \eta_i) \frac{P_{mi} Y_{mi}}{P_{nj} Y_{nj}} \pi^x_{j,mi} \pi^x_{nj,mi}] (\ln P_{mi} + \ln Y_{mi})$$

$$+ \sum_m \frac{\pi^f_{mj} \pi^f_{nmj} P_m \mathcal{F}_m}{P_{nj} Y_{nj}} \left((1 - \rho) \sum_o \pi^f_{omj} (\ln P_{oj}) - (1 - \rho) \sum_k \pi^f_{mk} \left[\sum_o \pi^f_{omk} (\ln P_{ok}) \right] \right)$$

$$+ \sum_m \frac{\pi^f_{mj} \pi^f_{nmj} P_m \mathcal{F}_m}{P_{nj} Y_{nj}} \left((1 - \gamma) \left(\ln P_{nj} - \sum_o \pi^f_{omj} (\ln P_{oj}) \right) \right)$$

$$+ \sum_m \frac{\pi^f_{mj} \pi^f_{nmj} P_m \mathcal{F}_m}{P_{nj} Y_{nj}} \frac{D_m WGDP}{P_m \mathcal{F}_m} \left(\left(\hat{D}_m - 1 \right) + \sum_o \sum_k \frac{\eta_k P_{ok} Y_{ok}}{WGDP} (\ln P_{ok} + \ln Y_{ok}) \right)$$

$$+ \sum_m \sum_i (1 - \eta_i) \frac{P_{mi} Y_{mi}}{P_{nj} Y_{nj}} \pi^x_{j,mi} \pi^x_{nj,mi} \left((1 - \varepsilon) \left(\sum_o \pi^x_{oj,mi} (\ln P_{oj}) \right) \right)$$

$$+ \sum_m \sum_i (1 - \eta_i) \frac{P_{mi} Y_{mi}}{P_{nj} Y_{nj}} \pi^x_{j,mi} \pi^x_{nj,mi} \left(- (1 - \varepsilon) \sum_k \pi^x_{k,mi,t} \sum_o \pi^x_{ok,mi} (\ln P_{ok}) \right)$$

$$+ \sum_m \sum_i (1 - \eta_i) \frac{P_{mi} Y_{mi}}{P_{nj} Y_{nj}} \pi^x_{j,mi} \pi^x_{nj,mi} \left((1 - \nu) \left(\ln P_{nj} - \sum_o \pi^x_{oj,mi} (\ln P_{oj}) \right) \right)$$

$$+ \sum_m \frac{\pi^f_{mj} \pi^f_{nmj} P_m \mathcal{F}_m}{P_{nj} Y_{nj}} [\ln \zeta_{mj} - \sum_k \pi^f_{mk} \ln \zeta_{mk}]$$

$$+ \sum_m \frac{\pi^f_{mj} \pi^f_{nmj} P_m \mathcal{F}_m}{P_{nj} Y_{nj}} [(1 - \rho) \sum_o \pi^f_{omj} \ln \tau^f_{omj} - (1 - \rho) \sum_k \pi^f_{mk} (\sum_o \pi^f_{omk} \ln \tau^f_{omk})]$$

$$+ \sum_m \frac{\pi^f_{mj} \pi^f_{nmj} P_m \mathcal{F}_m}{P_{nj} Y_{nj}} [(1 - \gamma) \ln \tau^f_{nmj} - (1 - \gamma) \sum_o \pi^f_{omj} \ln \tau^f_{omj}]$$

$$+ \sum_m \sum_i (1 - \eta_i) \frac{P_{mi} Y_{mi}}{P_{nj} Y_{nj}} \pi^x_{j,mi} \pi^x_{nj,mi} [\ln \vartheta_{j,mi} - \sum_k \pi^x_{k,mi} \ln \vartheta_{k,mi}]$$

$$+ \sum_m \sum_i (1 - \eta_i) \frac{P_{mi} Y_{mi}}{P_{nj} Y_{nj}} \pi^x_{j,mi} \pi^x_{nj,mi} [(1 - \varepsilon) \sum_o \pi^x_{oj,mi} \ln \tau^x_{oj,mi}]$$

$$+ \sum_m \sum_i (1 - \eta_i) \frac{P_{mi} Y_{mi}}{P_{nj} Y_{nj}} \pi^x_{j,mi} \pi^x_{nj,mi} [- (1 - \varepsilon) \sum_k \pi^x_{k,mi,t} \sum_o \pi^x_{ok,mi} \ln \tau^x_{ok,mi}]$$

$$+ \sum_m \sum_i (1 - \eta_i) \frac{P_{mi} Y_{mi}}{P_{nj} Y_{nj}} \pi^x_{j,mi} \pi^x_{nj,mi} [(1 - \nu) (\ln \tau^x_{nj,mi} - \sum_o \pi^x_{oj,mi} \ln \tau^x_{oj,mi})].$$

Define the following matrices:

- $\mathbf{\Psi^f}$ is a $NJ \times N$ matrix whose (nj, m)th element is $\frac{\pi^f_{mj} \pi^f_{nmj} P_m \mathcal{F}_m}{P_{nj} Y_{nj}}$, the share of nj's total revenue that comes from final sales to country m.

- Υ is a $N \times NJ$ matrix whose (m, mi)th element is $\frac{\eta_i P_{mi} Y_{mi}}{P_m \mathcal{F}_m}$, the share of value added of sector i in country m's GDP. Elements (n, mi) are 0 whenever $n \neq m$.

- $\mathbf{\Psi^x}$ is a $NJ \times NJ$ matrix whose $(nj, mi)th$ element is $\frac{\pi^x_{nj,mi} \pi^x_{j,mi} (1-\eta_i) P_{mi} Y_{mi}}{P_{nj,t} Y_{nj,t}}$, the share of country m, sector i's purchases from country n, sector j, in country n, sector j's total output.

- $\mathbf{\Pi^{1f}}$ is a $N \times NJ$ matrix whose $(m, nj)th$ element is $\pi^f_{mj} \pi^f_{nmj}$, the share of country n, sector j in country m's total consumption.

- $\mathbf{\Pi^{2f}}$ is a $N \times NJ$ matrix whose $(m, nj)th$ element is π^f_{nmj}, the share of country n in country m, sector j's spending.

- $\mathbf{\Pi^{1x}}$ is a $NJ \times NJ$ matrix whose $(nj, mi)th$ element is $\pi^x_{i,nj} \pi^x_{mi,nj}$, the share of country m, sector i in country n, sector j's total inputs.

- $\mathbf{\Pi^{2x}}$ is a $NJ \times NJ$ matrix whose $(mi, nj)th$ element is $\pi^x_{mi,nj}$.

- $\mathbf{\Psi^\zeta}$ a $NJ \times NJ$ matrix such that $\mathbf{\Psi^\zeta} = \mathbf{\Psi^{1\zeta}} + \mathbf{\Psi^{2\zeta}}$, where:

  - $\mathbf{\Psi^{1\zeta}}_{nj,mi} = \mathbf{\Psi^f}_{nj,m}$, and $\mathbf{\Psi^{1\zeta}}_{nj,mi} = 0, \forall i \neq j$
  - $\mathbf{\Psi^{2\zeta}}_{nj,mj} = -\mathbf{\Psi^f}_{nj,m} \pi^f_{mk}$

- $\mathbf{\Psi^{\tau^f}}$ a $NJ \times NNJ$ matrix such that $\mathbf{\Psi^{\tau^f}} = \mathbf{\Psi^{1\tau^f}} + \mathbf{\Psi^{2\tau^f}} + \mathbf{\Psi^{3\tau^f}}$, where:

  - $\mathbf{\Psi^{1\tau^f}}_{nj,nmj} = (1-\gamma) \mathbf{\Psi^f}_{nj,m}$, and $\mathbf{\Psi^{1\tau^f}}_{nj,omi} = 0, \forall i \neq j$ or $n \neq o$
  - $\mathbf{\Psi^{2\tau^f}}_{nj,omj} = [(1-\rho) - (1-\gamma)] \mathbf{\Psi^f}_{nj,m} \pi^f_{omj}$, and $\mathbf{\Psi^{2\tau^f}}_{nj,omi} = 0, \forall j \neq i$
  - $\mathbf{\Psi^{3\tau^f}}_{nj,omi} = -(1-\rho) \mathbf{\Psi^f}_{nj,m} \pi^f_{mi} \pi^f_{omi}$

- $\mathbf{\Psi^\vartheta}$ a $NJ \times NJJ$ matrix such that $\mathbf{\Psi^\vartheta} = \mathbf{\Psi^{1\vartheta}} + \mathbf{\Psi^{2\vartheta}}$, where:

  - $\mathbf{\Psi^{1\vartheta}}_{nj,mik} = \mathbf{\Psi^x}_{nj,mi}$, and $\mathbf{\Psi^{1\vartheta}}_{nj,mik} = 0, \forall j \neq k$
  - $\mathbf{\Psi^{2\vartheta}}_{nj,mik} = -\mathbf{\Psi^x}_{nj,mi} \pi^x_{k,mi}$

- $\mathbf{\Psi^{\tau^x}}$ a $NJ \times NJNJ$ matrix such that $\mathbf{\Psi^{\tau^x}} = \mathbf{\Psi^{1\tau^x}} + \mathbf{\Psi^{2\tau^x}} + \mathbf{\Psi^{3\tau^x}}$, where:

  - $\mathbf{\Psi^{1\tau^x}}_{nj,njmi} = (1-\nu) \mathbf{\Psi^x}_{nj,mi}$, and $\mathbf{\Psi^{1\tau^x}}_{nj,okmi} = 0, \forall n \neq o$ or $k \neq j$
  - $\mathbf{\Psi^{2\tau^x}}_{nj,ojmi} = [(1-\varepsilon) - (1-\nu)] \mathbf{\Psi^x}_{nj,mi} \pi^x_{oj,mi}$, and $\mathbf{\Psi^{2\tau^x}}_{nj,okmi} = 0, \forall j \neq k$
  - $\mathbf{\Psi^{3\tau^x}}_{nj,okmi} = -(1-\varepsilon) \mathbf{\Psi^x}_{nj,mi} \pi^x_{k,mi,t} \pi^x_{ok,mi}$

- $\mathbf{\Psi^\Delta}$ a $(NJ) \times N$ matrix, where $\Psi^\Delta_{nj,m} = \frac{\pi^f_{mj} \pi^f_{nmj} P_m \mathcal{F}_m}{P_{nj} Y_{nj}} \frac{D_m WGDP}{P_m \mathcal{F}_m}$

- $\mathbf{\Psi^{PY\Delta}}$, where $\Psi^{PY\Delta}_{nj,ok} = \frac{\eta_k P_{ok} Y_{ok}}{WGDP} \sum_m \Psi^\Delta_{nj,m}$.

Using the matrices defined above, the market clearing conditions for all countries and sectors can be grouped and written in matrix form as:

$$\ln \mathbf{P} + \ln \mathbf{Y} = \left(\mathbf{\Psi}^f \mathbf{\Upsilon} + \mathbf{\Psi}^x + \mathbf{\Psi}^{PY\Delta} \right) (\ln \mathbf{P} + \ln \mathbf{Y})$$
$$+ \left[(1 - \gamma) \, diag \left(\mathbf{\Psi}^f \mathbf{1} \right) + [(1 - \rho) - (1 - \gamma)] \, \mathbf{\Psi}^f \mathbf{\Pi}^{2f} - (1 - \rho) \, \mathbf{\Psi}^f \mathbf{\Pi}^{1f} \right] \ln \mathbf{P} \qquad \text{(A.4)}$$
$$+ \left[(1 - \nu) \, diag \left(\mathbf{\Psi}^x \mathbf{1} \right) + [(1 - \varepsilon) - (1 - \nu)] \, \mathbf{\Psi}^x \mathbf{\Pi}^{2x} - (1 - \varepsilon) \, \mathbf{\Psi}^x \mathbf{\Pi}^{1x} \right] \ln \mathbf{P}$$
$$+ \mathbf{\Psi}^\zeta \ln \zeta + \mathbf{\Psi}^{\tau f} \ln \tau^f + \mathbf{\Psi}^\vartheta \ln \vartheta + \mathbf{\Psi}^{\tau x} \ln \tau^x + \mathbf{\Psi}^\Delta \Delta.$$

We can then solve for the vector of prices as a function of output and the exogenous shocks:

$$\ln \mathbf{P} = \mathcal{P}^Y \ln \mathbf{Y} + \mathcal{P}^\zeta \ln \zeta + \mathcal{P}^{\tau f} \ln \tau^f + \mathcal{P}^\vartheta \ln \vartheta + \mathcal{P}^{\tau x} \ln \tau^x + \mathcal{P}^\Delta \Delta \qquad \text{(A.5)}$$

where

$$\mathcal{P}^Y = -(\mathbf{I} - \mathcal{M})^+ \left(\mathbf{I} - \mathbf{\Psi}^f \mathbf{\Upsilon} - \mathbf{\Psi}^x - \mathbf{\Psi}^{PY\Delta} \right),$$

for the other shocks $s \in \{\zeta, \tau^f, \vartheta, \tau^x, \Delta\}$

$$\mathcal{P}^s = -(\mathbf{I} - \mathcal{M})^+ \mathbf{\Psi}^s,$$

and

$$\mathcal{M} = \mathbf{\Psi}^f \mathbf{\Upsilon} + \mathbf{\Psi}^x + \mathbf{\Psi}^{PY\Delta} + \left[(1 - \gamma) \, diag \left(\mathbf{\Psi}^f \mathbf{1} \right) + [(1 - \rho) - (1 - \gamma)] \, \mathbf{\Psi}^c \mathbf{\Pi}^{2f} - (1 - \rho) \, \mathbf{\Psi}^f \mathbf{\Pi}^{1f} \right]$$
$$+ \left[(1 - \nu) \, diag \left(\mathbf{\Psi}^x \mathbf{1} \right) + [(1 - \varepsilon) - (1 - \nu)] \, \mathbf{\Psi}^x \mathbf{\Pi}^{2x} - (1 - \varepsilon) \, \mathbf{\Psi}^x \mathbf{\Pi}^{1x} \right].$$

A.2.2 Labor Block

Substituting (2.36) and (2.37) into (2.35):

$$\frac{1}{\psi} \ln L_n = \sum_j \pi_{nj}^H \left(\ln P_{nj} + \ln Y_{nj} - \ln H_{nj} \right) - \ln P_n.$$

Substituting $\ln H_{nj}$ from (2.34):

$$\frac{1}{\psi} \ln L_n = \sum_j \pi_{nj}^H \left(\ln P_{nj} + \ln Y_{nj} - \left[\ln L_n + \frac{\mu - 1}{\mu} \ln \pi_{nj}^H \right] \right) - \ln P_n.$$

Take $\ln L_n$ on the left side and use the fact that $\sum_j \pi_{nj} \ln \pi_{nj} = 0$:

$$\frac{1}{\psi} \ln L_n + \ln L_n = \sum_j \pi_{nj}^H \left(\ln P_{nj} + \ln Y_{nj} \right) - \ln P_n$$

$$\ln L_n = \frac{\psi}{1 + \psi} \left[\sum_j \pi_{nj}^H \left(\ln P_{nj} + \ln Y_{nj} \right) - \ln P_n \right]. \tag{A.6}$$

To solve for $\ln H_{nj}$ as a function of prices and output, start from (2.34), and use (2.37) and (2.38) to write

$$\ln H_{nj} = \ln L_n + \frac{\mu - 1}{\mu} \left(\ln P_{nj} + \ln Y_{nj} - \sum_k \pi_{nk}^H \left(\ln P_{nk} + \ln Y_{nk} \right) \right). \tag{A.7}$$

A.2.3 Price Block

Plugging (A.7) and then (A.6) into (2.31) leads to:

$$\ln P_{nj} = -\ln Z_{nj} + \eta_j \left(-\left(1 - \alpha_j\right) \frac{\mu - 1}{\mu} \right) \left(\ln P_{nj} + \ln Y_{nj} \right)$$

$$+ \left(1 - \alpha_j\right) \eta_j \frac{\psi}{1 + \psi} \ln P_n + \left(1 - \eta_j\right) \ln P_{nj}^X$$

$$+ \left(1 - \alpha_j\right) \eta_j \left(\frac{\mu - 1}{\mu} - \frac{\psi}{1 + \psi} \right) \sum_k \pi_{nk}^H \left(\ln P_{nk} + \ln Y_{nk} \right).$$

In turn, plugging in (2.32) and (2.33) leads to:

$$\ln P_{nj} = -\left(\ln Z_{nj} + \left(1 - \eta_j\right) \sum_i \pi_{i,nj}^x \frac{\ln \vartheta_{i,nj}}{1 - \varepsilon} \right) \tag{A.8}$$

$$+ \left(\eta_j - \left(1 - \alpha_j\right) \eta_j \frac{\mu - 1}{\mu} \right) \left(\underbrace{\ln P_{nj} + \ln Y_{nj}}_{\mathbf{A^1}} \right)$$

$$+ \left(\left(1 - \alpha_j\right) \eta_j \frac{\mu - 1}{\mu} - \frac{\left(1 - \alpha_j\right) \eta_j \psi}{1 + \psi} \right) \left[\sum_k \pi_{nk}^H \left[\underbrace{\ln P_{nk} + \ln Y_{nk}}_{\mathbf{A^2}} \right] \right]$$

$$+ \frac{\psi}{1 + \psi} \left(1 - \alpha_j\right) \eta_j \sum_k \pi_{nk}^f \left[\frac{1}{1 - \rho} \left(\underbrace{\ln \zeta_{nk}}_{\mathbf{B}} \right) + \sum_m \pi_{mnk}^f \left(\underbrace{\ln \tau_{mnk}^f}_{\mathbf{C}} + \underbrace{\ln P_{mk}}_{\mathbf{D}} \right) \right]$$

$$+ \left(1 - \eta_j\right) \sum_i \pi_{i,nj}^x \left[\sum_m \pi_{mi,nj}^x \left(\underbrace{\ln \tau_{mi,nj}^x}_{\mathbf{F}} + \underbrace{\ln P_{mi}}_{\mathbf{G}} \right) \right],$$

where the matrices in the horizontal curly brackets denote the matrices multiplying those objects, rather than the objects themselves. Put in matrix form:

$$\ln \mathbf{P} = -\ln \mathbf{Z} + \mathbf{A} \left(\ln \mathbf{P} + \ln \mathbf{Y} \right) + \mathbf{B} \ln \zeta + \mathbf{C} \ln \tau^f + \mathbf{D} \ln \mathbf{P} + \mathbf{F} \ln \tau^x + \mathbf{G} \ln \mathbf{P} \tag{A.9}$$

where

- $\mathbf{A} = \mathbf{A}^1 + \mathbf{A}^2$ where

  1. $\mathbf{A}^1_{nj,nj} = \eta_j - (1 - \alpha_j)\,\eta_j \frac{\mu-1}{\mu}$, 0 off-diagonal
  2. $\mathbf{A}^2_{nj,nk} = \left(\frac{\mu-1}{\mu} - \frac{\psi}{1+\psi} \right)(1 - \alpha_j)\,\eta_j \pi^H_{nk}$, 0 for different country $(\mathbf{A}^2_{nj,mk} = 0, \forall n \neq m)$

- $\mathbf{B}_{nj,nk} = \eta_j \frac{(1-\alpha_j)\psi}{1+\psi} \frac{1}{1-\rho} \pi^f_{nk}$, 0 for different country $(B_{nj,mk} = 0, \forall n \neq m)$

- $\mathbf{C}_{nj,mnk} = \eta_j \frac{(1-\alpha_j)\psi}{1+\psi} \pi^f_{nk} \pi^f_{mnk}$, 0 for different country $(C_{nj,mok} = 0, \forall n \neq o)$

- $\mathbf{D}_{nj,mk} = \eta_j \frac{(1-\alpha_j)\psi}{1+\psi} \pi^f_{nk} \pi^f_{mnk}$

- $\mathbf{F}_{nj,minj} = (1 - \eta_j)\,\pi^x_{i,nj} \pi^x_{mi,nj}$, 0 for different country-sector $(F_{nj,miok} = 0, \forall (nj) \neq (ok))$

- $\mathbf{G}_{nj,mi} = (1 - \eta_j)\,\pi^x_{i,nj} \pi^x_{mi,nj}$.

Note that in stating (A.9), we abstracted from the aggregated taste shift, $(1 - \eta_j) \sum_i \pi^x_{i,nj} \frac{\ln \vartheta_{i,nj}}{1-\varepsilon}$, in (A.8). A net increase in $\vartheta_{i,nj}$ is isomorphic to an increase in productivity Z_{nj}. Thus, users interested in net changes in $\vartheta_{i,nj}$ across all sectors i should simulate changes in Z_{nj}. The code will automatically renormalize any vector of $\ln \vartheta_{i,nj}$ specified by the user so that they have a zero net productivity impact, and represent purely a *relative* shift in preferences across sectors. This can be seen most clearly in (2.29), which shows that the change in the expenditure share is a function of only the renormalized taste shifter, $\ln \vartheta_{i,nj} - \sum_k \pi^x_{k,nj,t} \ln \vartheta_{k,nj}$.

A.2.4 Influence for Output

Plugging equation (A.9) into (A.5) provides the influence matrix for output:

$$
\begin{aligned}
\ln \mathbf{Y} = \ \ & \underbrace{\left[\mathbf{A} - (\mathbf{I} - \mathbf{A} - \mathbf{D} - \mathbf{G})\,\mathcal{P}^{\mathcal{Y}}\right]^{-1}}_{\Lambda^Y_Z} \bigg\{ \ln \mathbf{Z} + \left[(\mathbf{I} - \mathbf{A} - \mathbf{D} - \mathbf{G})\,\mathcal{P}^\zeta - \mathbf{B}\right] \ln \zeta \\
& + \left[(\mathbf{I} - \mathbf{A} - \mathbf{D} - \mathbf{G})\,\mathcal{P}^{\tau^f} - \mathbf{C}\right] \ln \tau^f + \left[(\mathbf{I} - \mathbf{A} - \mathbf{D} - \mathbf{G})\,\mathcal{P}^{\tau^x} - \mathbf{F}\right] \ln \tau^x \\
& + (\mathbf{I} - \mathbf{A} - \mathbf{D} - \mathbf{G})\,\mathcal{P}^\vartheta \ln \vartheta + (\mathbf{I} - \mathbf{A} - \mathbf{D} - \mathbf{G})\,\mathcal{P}^\Delta \Delta \bigg\}.
\end{aligned} \tag{A.10}
$$

This equation characterizes the influence matrices in equation (2.39).

A.2.5 Influence for Hours

Taking the log deviation of equation (2.6) and plugging in for the log deviation in W_n from (2.36) and (2.37) gives:

$$\ln H_{nj} = -\psi \ln P_n + (\mu - 1)\ln W_{nj} + (\psi - \mu + 1)\left(\sum_k \pi^H_{nk}\left(\ln P_{nk} + \ln Y_{nk} - \ln H_{nk}\right)\right).$$

Using the firms' optimal labor choice and substituting in for the sectoral wage $\ln W_{nj}$ (2.17) yields:

$$\ln H_{nj} = -\psi \ln P_n + (\mu - 1)\left(\ln P_{nj} + \ln Y_{nj} - \ln H_{nj}\right)$$
$$+ (\psi - \mu + 1)\left(\sum_k \pi^H_{nk}\left(\ln P_{nk} + \ln Y_{nk} - \ln H_{nk}\right)\right),$$

Rearranging and plugging in for $\ln P_n$ from (2.33) leads to:

$$\mu \ln H_{nj} = (\psi - \mu + 1)\sum_k \pi^H_{nk}\left(\underbrace{\ln P_{nk}}_{\mathbf{B}^H_1} + \underbrace{\ln Y_{nk}}_{\mathbf{C}^H_1} - \underbrace{\ln H_{nk}}_{\mathbf{D}^H}\right)$$
$$-\psi \sum_k \pi^f_{nk}\left[\underbrace{\frac{\ln \zeta_{nk}}{1-\rho}}_{\mathbf{E}^H} + \sum_m \pi^f_{mnk}\left(\underbrace{\ln \tau^f_{mnk}}_{\mathbf{F}^H} + \underbrace{\ln P_{mk}}_{\mathbf{B}^H_2}\right)\right] + (\mu - 1)\left(\underbrace{\ln P_{nj} + \ln Y_{nj}}_{\mathbf{C}^H_2}\right),$$

where the matrices in the horizontal curly brackets denote the matrices multiplying those objects, rather than the objects themselves. In matrix form:

$$\ln \mathbf{H} = \left(\mu \mathbf{I} + \mathbf{D}^H\right)^{-1}\left(\mathbf{B}^H \ln \mathbf{P} + \mathbf{C}^H \ln \mathbf{Y} + \mathbf{E}^H \ln \zeta + \mathbf{F}^H \ln \tau^f\right) \qquad (A.11)$$

where:

1. $\mathbf{B}^H = \mathbf{B}^H_1 + \mathbf{B}^H_2 + (\mu - 1)\,\mathbf{I}_{NJxNJ}$, where $\mathbf{B}^H_{1,nj,nk} = (\psi - \mu + 1)\,\pi^H_{nk}$ (block diagonal) and $\mathbf{B}^H_{2,nj,mk} = -\psi \pi^f_{nk}\pi^f_{mnk}$

2. $\mathbf{C}^H = \mathbf{C}^H_1 + \mathbf{C}^H_2$, where $\mathbf{C}^H_{1,nj,nk} = (\psi - \mu + 1)\,\pi^H_{nk}$ (block diagonal) and $\mathbf{C}^H_{2,nj,nj} = \mu - 1$ (0 off diagonal)

3. $\mathbf{D}^H_{nj,nk} = (\psi - \mu + 1)\,\pi^H_{nk}$ (block diagonal)

4. $\mathbf{E}^H_{nj,nk} = -\frac{\psi}{1-\rho}\pi^f_{nk}$ (block diagonal)

5. $\mathbf{F}^H_{nj,mnk} = -\psi \pi^f_{nk}\pi^f_{mnk}$ ($\mathbf{F}^H_{nj,mlk} = 0$ if $n \neq l$)

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6. and $\ln \mathbf{P}$ and $\ln \mathbf{Y}$ are solved from their respective influence matrices.

Equation (A.11) gives the functional form for the influence matrices in (2.40).

A.2.6 Influence for Aggregate Labor

Substituting (2.36), (2.37), and (2.33) into (2.35):

$$\ln L_n = \frac{\psi}{1+\psi} \sum_j \pi_{nj}^H \left(\underbrace{\ln P_{nj}}_{\mathbf{B}_1^L} + \underbrace{\ln Y_{nj}}_{\mathbf{C}^L} \right) - \frac{\psi}{1+\psi} \sum_j \pi_{nj}^f \left[\underbrace{\frac{\ln \zeta_{nk}}{1-\rho}}_{\mathbf{E}^L} + \sum_m \pi_{mnj}^f \left(\underbrace{\ln \tau_{mnj}^f}_{\mathbf{F}^L} + \underbrace{\ln P_{mj}}_{\mathbf{B}_2^L} \right) \right].$$

In matrix form:

$$\ln \mathbf{L} = \mathbf{B}^L \ln \mathbf{P} + \mathbf{C}^L \ln \mathbf{Y} + \mathbf{E}^L \ln \zeta + \mathbf{F}^L \ln \tau^f$$

where:

- $\mathbf{B}^L = \mathbf{B}_1^L + \mathbf{B}_2^L$, where $\mathbf{B}_{1,n,nj}^L = \frac{\psi}{1+\psi}\pi_{nj}^H$ and $\mathbf{B}_{2,n,mj}^L = -\frac{\psi}{1+\psi}\pi_{nj}^f \pi_{mnj}^f$

- $\mathbf{C}_{n,nj}^L = \frac{\psi}{1+\psi}\pi_{nj}^H$

- $\mathbf{E}_{n,nj}^L = -\frac{\psi}{1+\psi}\frac{1}{1-\rho}\pi_{nj}^f$

- $\mathbf{F}_{n,mnj}^L = -\frac{\psi}{1+\psi}\pi_{nj}^f \pi_{mnj}^f$.

Plugging in (A.5) for $\ln \mathbf{P}$ and in turn (2.39) for $\ln \mathbf{Y}$ yields the change in the aggregate labor input following the shocks. Note that the $\ln \mathbf{L}$ is outputted by the linear model solution code but not by the exact hat algebra code.

A.3 The Exact Hat Equations and Solution Algorithm

As above, denote by a ' (a prime) a value of a variable post-shock. The pre-shock ("steady state") value of the variable is written without a prime. Denote by ^ a gross proportional change between the post-shock value and the pre-shock value: $\forall x$, $\hat{x} = \frac{x'}{x}$.

Then, the exact hat equilibrium changes in output $\hat{Y}_{nj}$, prices $\hat{P}_{nj}$, sectoral final consumption shares $\hat{\pi}_{mj}^f$, final trade shares $\hat{\pi}_{nmj}^f$, sectoral intermediate usage shares $\hat{\pi}_{j,mi}^x$, intermediate trade shares $\hat{\pi}_{nj,mi}^x$, sectoral wages $\hat{W}_{nj}$ are as follows:

Market clearing conditions.

$$\left(\hat{P}_{nj}\hat{Y}_{nj} \right) \left(P_{nj}Y_{nj} \right) = \sum_m \left(\sum_i \eta_i \left(\hat{P}_{mi}\hat{Y}_{mi} \right) \left(P_{mi}Y_{mi} \right) + \hat{D}_m D_m \right) \pi_{mj}^f \hat{\pi}_{mj}^f \pi_{nmj}^f \hat{\pi}_{nmj}^f \qquad (A.12)$$

$$+ \sum_m \sum_i (1-\eta_i) \left(\hat{P}_{mi}\hat{Y}_{mi} \right) \left(P_{mi}Y_{mi} \right) \pi_{j,mi}^x \hat{\pi}_{j,mi}^x \hat{\pi}_{nj,mi}^x \pi_{nj,mi}^x.$$

$$\hat{W}_{nj} \left(\frac{\hat{W}_n}{\hat{P}_n} \right)^{\psi} \left(\frac{\hat{W}_{nj}}{\hat{W}_n} \right)^{\mu-1} = \hat{P}_{nj} \hat{Y}_{nj} \tag{A.13}$$

where

$$\hat{W}_n = \left(\sum \pi_{nj}^H \hat{W}_{nj}^{\mu} \right)^{\frac{1}{\mu}}. \tag{A.14}$$

Prices and price indices.

$$\begin{aligned} \hat{P}_{nj} &= \hat{Z}_{nj}^{-1} \hat{W}_{nj}^{(1-\alpha_j)\eta_j} \hat{R}_{nj}^{\alpha_j \eta_j} \left(\hat{P}_{nj}^X \right)^{1-\eta_j} \\ &= \hat{Z}_{nj}^{-1} \hat{W}_{nj}^{(1-\alpha_j)\eta_j} \left(\hat{P}_{nj} \hat{Y}_{nj} \right)^{\alpha_j \eta_j} \left(\hat{P}_{nj}^X \right)^{1-\eta_j} \end{aligned} \tag{A.15}$$

$$\hat{P}_{nj}^f = \left[\sum_m \pi_{mnj}^f (\hat{\tau}_{mnj}^f \hat{P}_{mj})^{1-\gamma} \right]^{\frac{1}{1-\gamma}} \tag{A.16}$$

$$\hat{P}_n = \left[\sum_j \pi_{nj,t}^f \hat{\zeta}_{nj} (\hat{P}_{nj}^f)^{1-\rho} \right]^{\frac{1}{1-\rho}} \tag{A.17}$$

$$\hat{P}_{i,nj}^X = \left[\sum_i \pi_{mi,nj,t}^x \left(\hat{\tau}_{mi,nj}^x \hat{P}_{mi} \right)^{1-\nu} \right]^{\frac{1}{1-\nu}} \tag{A.18}$$

$$\hat{P}_{nj}^X = \left[\sum_i \pi_{i,nj,t}^x \hat{\vartheta}_{i,nj} \left(\hat{P}_{i,nj}^X \right)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}. \tag{A.19}$$

Expenditure shares.

$$\hat{\pi}_{nj}^f = \frac{\hat{\zeta}_{nj} \left(\hat{P}_{nj}^f \right)^{1-\rho}}{\sum_k \pi_{nk}^f \hat{\zeta}_{nk} \left(\hat{P}_{nk}^f \right)^{1-\rho}} \tag{A.20}$$

$$\hat{\pi}_{mnj}^f = \frac{\left(\hat{\tau}_{mnj}^f \hat{P}_{mj} \right)^{1-\gamma}}{\sum_k \pi_{knj}^f \left(\hat{\tau}_{knj}^f \hat{P}_{kj} \right)^{1-\gamma}} \tag{A.21}$$

$$\hat{\pi}_{i,nj}^x = \frac{\hat{\vartheta}_{i,nj} \left(\hat{P}_{i,nj}^X \right)^{1-\varepsilon}}{\sum_k \pi_{k,nj}^x \hat{\vartheta}_{k,nj} \left(\hat{P}_{k,nj}^X \right)^{1-\varepsilon}} \tag{A.22}$$

$$\hat{\pi}_{mi,nj}^x = \frac{\left(\hat{\tau}_{mi,nj}^x \hat{P}_{mj} \right)^{1-\nu}}{\sum_k \pi_{ki,nj}^x \left(\hat{\tau}_{ki,nj}^x \hat{P}_{kj} \right)^{1-\nu}}. \tag{A.23}$$

Algorithm.   The code implements the following algorithm to solve this system of equations:

1. Specify the shocks $\hat{Z}_{nj}$, $\hat{\zeta}_{nj}$, $\hat{\vartheta}_{i,nj}$, $\hat{\tau}^f_{mni}$, $\hat{\tau}^x_{mi,nj}$ and $\hat{D}_m$ $\forall n, m, j, i$.

2. Guess $\hat{P}_{nj}$.

3. Given the guess in **2.**, evaluate $\hat{P}^f_{nj}$, $\hat{P}_{nj}$, $\hat{P}^X_{i,nj}$, and $\hat{P}^X_{nj}$ using (A.16), (A.17), (A.18), and (A.19). Given these, evaluate $\hat{\pi}^f_{nj}$, $\hat{\pi}^f_{mnj}$, $\hat{\pi}^x_{i,nj}$, and $\hat{\pi}^x_{mi,nj}$ using (A.20), (A.21), (A.22), and (A.23).

4. Solve for $\hat{P}_{nj}\hat{Y}_{nj}$ using the new trade shares in **3.** and (A.12). Since the solution is unique up to a numeraire, we further impose that the world GDP remains constant: $\sum_{m,i} \eta_i \left(\hat{P}_{mi}\hat{Y}_{mi} \right) (P_{mi}Y_{mi}) = \sum_{m,i} \eta_i (P_{mi}Y_{mi})$ when solving for $\hat{P}_{nj}\hat{Y}_{nj}$. In the code we also reexpress the initial deficit D_m as a share of the (constant) world GDP.

5. Solve for the wages $\hat{W}_{nj}$ using (A.13)-(A.14).

6. Evaluate the new guess for prices $\hat{P}_{nj}$ using (A.15).

7. Go back to **2.** Iterate to convergence.

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